Math 255A Lecture 22 Notes

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1 Riesz Projection and Spectra of Self-Adjoint Operators

1.1 Riesz projection

Let $T : B \to B$ be compact and $\lambda \in \operatorname{Spec}(T) \setminus \{0\}$. Then $(T - (\lambda + z)I)^{-1} = \sum_{j=-N}^{\infty} A_j z^j$ for some $1 \leq N < \infty$.

Proposition 1.1 (Riesz). *The operator $-A_{-1}$ is a projection onto the finite dimensional generalized eigenpace $N_\lambda = \bigcup_{k=1}^{\infty} \ker(T - \lambda I)^k$.*

Proof. Multiply the Laurent expansion by z^{-j-1} for $-N \leq -1 \leq -1$, and integrate over $\partial D(0, r)$ with $0 < r \ll 1$. We get

$$A_j = \frac{1}{2\pi i} \int_{\partial D(0,r)} (T - (\lambda + z)I)^{-1} z^{-j-1} \, dz,$$

so we get

$$\Pi = -A_{-1} = \frac{1}{2\pi i} \int_{\partial D(0,r)} ((\lambda + z)I - T)^{-1} \, dz.$$

We now claim that Π is a projection. Let $0 < r_1 < r_2 \ll 1$, and write

$$\begin{aligned}
\Pi^2 &= \int_{\partial D(0,r_2)} \int_{\partial D(0,r_1)} ((\lambda + w)I - T)^{-1}((\lambda + z)I - T)^{-1} \frac{1}{2\pi i} \, dz \frac{1}{2\pi i} \, dw \\
&= \int_{\partial D(0,r_2)} \int_{\partial D(0,r_1)} \frac{1}{w - z}((\lambda + z)I - T)^{-1} \frac{1}{2\pi i} \, dz \frac{1}{2\pi i} \, dw \\
&\quad - \int_{\partial D(0,r_2)} \int_{\partial D(0,r_1)} \frac{1}{w - z}((\lambda + w)I - T)^{-1} \frac{1}{2\pi i} \, dz \frac{1}{2\pi i} \, dw
\end{aligned}$$

Apply Cauchy's integral formula to both terms. The second term equals 0.

$$= \Pi.$$

Now in the Laurent expansion, multiply by $T-(\lambda+z)I$ on the left to get

$$I=(T-\lambda I)A_{-N}z^{-N}+\sum_{j=-N+1}^{\infty}((T-\lambda I)A_j-A_{j-1})z^j,$$

which gives

$$(T-\lambda I)A_{-N}=A_{-N}(T-\lambda I)=0$$
$$(T-\lambda I)A_j-A_{j-1}=\begin{cases}0 & j\neq 0, j\geq N+1\\ 1 & j=0.\end{cases}$$

So $[T,A_j]=0$ for all j, and

$$A_{-N}=(T-\lambda I)A_{-N+1}=(T-\lambda I)^2A_{-N+2}=\cdots=(T-\lambda I)^{N-I}A_{-1}.$$

We get that $(T-\lambda I)^NA_{-1}=0$. Also, $I+A_{-1}=(T-\lambda I)A_0$, so applying $(T-\lambda I)^N$ gives us

$$(T-\lambda I)^N=A_0(T-\lambda I)^{N+1}.$$

Thus, if $(T-\lambda I)^{N+1}x=0$, then $(T-\lambda I)^Nx=0$. It follows that $N_\lambda=\ker(T-\lambda I)^N$, so $\dim(\ker(T_\lambda))<\infty$ because $T-\lambda I$ is Fredholm of index 0.

It remains to show that $\mathrm{im}(A_{-1})=N_\lambda=\ker((T-\lambda I)^N)$. If $x\in N_\lambda$, then $x+A_{-1}x=(T-\lambda I)A_0x=(T_\lambda I)^2A_1x=\cdots=(T-\lambda I)^NA_{N-1}x=0$. So $\mathrm{im}(A_{-1})=N_\lambda$. □

We can write $B=N_\lambda\oplus\ker(\Pi)$. This is a T-invariant decomposition. Moreover, $(T-\lambda I)|_{N_\lambda}$ is nilpotent, and $(T-\lambda I)_{\ker(\Pi)}$ is bijective.

1.2 Spectra of self-adjoint operators

Assume now that $B=H$ is a complex Hilbert space.

Definition 1.1. An operator is **self-adjoint** if $\langle Tx,y\rangle=\langle x,Ty\rangle$ for all $x,y\in H$.

Example 1.1. Let $H=L^2((0,1))$, and let $Tu(x)=\int_0^1K(x,y)u(y)\,dy$, where $K\in C([0,1]\times[0,1])$ is such that $\overline{K(x,y)}=K(y,x)$.

Proposition 1.2. *Let $T\in\mathcal{L}(H,H)$ be self-adjoint. Then $\mathrm{Spec}(T)\subseteq\mathbb{R}$, and the resolvent $R(z)=(T-zI)^{-1}\in\mathcal{L}(H,H)$ satisfies $\|R(z)\|_{\mathcal{L}(H,H)}\leq 1/|\operatorname{Im}(z)|$ for $z\in\mathbb{C}\setminus\mathbb{R}$.*

Proof. Let $z=i+iy$ with $y\neq 0$, and compute

$$\begin{aligned}\|(T-zI)u\|^2&=\langle(T-xI)u-iyu,(T-xI)u-iyu\rangle\\&=\|(T-x)u\|^2+\underbrace{i\,\langle((T-x)u,yu\rangle-i\,\langle yu,(T-x)u\rangle}_{=0}+y^2\|u\|^2.\end{aligned}$$

We get

$$\|(T-z)u\|^2 = \|(T-x)u\|^2 + y^2\|u\|^2 \geq y^2\|u\|^2,$$

so $\|(T-zI)u\| \geq |\operatorname{Im}(z)|\|u\|$, so $T-zI$ is injective and $\operatorname{im}(T-zI)$ is closed. So $H = \operatorname{im}(T-z) \oplus \operatorname{im}(T-z)^\perp$, where $\operatorname{im}(T-z)^\perp = \{x : \langle (T-z)y, x\rangle = 0 \,\forall y \in H\} = \ker(T-\overline{z}I) = \{0\}$. So we get that $T - zI : H \to H$ is bijective, and $\|(T-z)^{-1}\|_{\mathcal{L}(H,H)} \leq 1/|\operatorname{Im}(z)|$. □

Remark 1.1. Let $T \in \mathcal{L}(H,H)$. Then T is uniquely determined by the function $x \mapsto \langle Tx, x\rangle$. If $\langle Tx, x\rangle = 0$ for all x, then we polarize:

$$\langle T(y+z), y+z\rangle = 0, \qquad \langle T(y+iz), y+iz\rangle = 0,$$

for all $y, z \in H$, so

$$\langle Ty, z\rangle + \langle Tz, y\rangle = 0, \qquad \langle Ty, z\rangle - \langle Tz, y\rangle = 0,$$

which give us $\langle Ty, z\rangle = 0$. So $T = 0$. So T is self adjoint if and only if $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in H$.

Now let T be compact and self adjoint. Let $\lambda \in \operatorname{Spec}(T) \setminus \{0\}$. Then $z \mapsto (T-zI)^{-1}$ has a pole at $z = \lambda$, and the pole is simple. We get

$$(T-zI)^{-1} = \frac{\Pi\lambda}{\lambda - z} + \operatorname{Hol}(z)$$

for $0 < |z-\lambda| \ll 1$. $\Pi\lambda$ is projection onto $\ker(T-\lambda I)$, and Π_λ is self-adjoint. Indeed, $\Pi_\lambda = \lim_{z\to\lambda}(\lambda - z)(T-zI)^{-1}$, and if z approaches λ along the real axis, then this is self-adjoint.

Next time, we will show that

$$(T-zI)^{-1} = \sum_{\lambda_j \in \operatorname{Spec}(T)\setminus\{0\}} \frac{\Pi\lambda_j}{\lambda_j - z}$$

for $\operatorname{Im}(z) \neq 0$.